The Borel complexity of the bi-interpretability relation between omega-categorical structures

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Generic structures
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Results, and a question (will explain...)

Theorem (N, Schlicht and Tent, J. Math Logic 2021)

BI is Borel-below E_∞, where BI is the bi-interpretability relation between omega-categorical structures, and E_∞ is a Borel equivalence relation with all classes countable.

Recall that $G \leq_c \mathrm{Sym}(\mathbb{N})$ is called oligomorphic if for each k, the action of G on $\mathbb{N}^k$ only has finitely many orbits.

Corollary

The topological isomorphism relation between oligomorphic groups is also Borel-below E_∞.

Question

Is there a lower bound other than $id_{\mathbb{R}}$ on the complexity?

Borel classes of closed subgroups of Sym($\mathbb{N}$)

The closed subgroups G of $\mathrm{Sym}(\mathbb{N})$ form a "standard Borel space":

- If σ is a string let $[\sigma] = \{\pi \in \mathrm{Sym}(\mathbb{N}) \colon \sigma \prec \pi\}$.
- The σ-algebra of Borel sets is generated by the sets

$$\{G \colon G \cap [\sigma] \neq \emptyset\}.$$

Programme (Kechris, N. and Tent, 2018; Logic Blog 2020)

(a) Determine whether classes $\mathcal{C}$ of closed subgroups of S_∞ are Borel.

(b) If $\mathcal{C}$ is Borel, study the relative complexity of the topological isomorphism relation, using Borel reducibility $\leq_B$.

Largest $\mathcal{C}$: locally Roelcke precompact groups

By G we always denote a closed subgroup of $\mathrm{Sym}(\mathbb{N})$.
Note that G is compact iff each open subgroup has only finitely many (left) cosets.

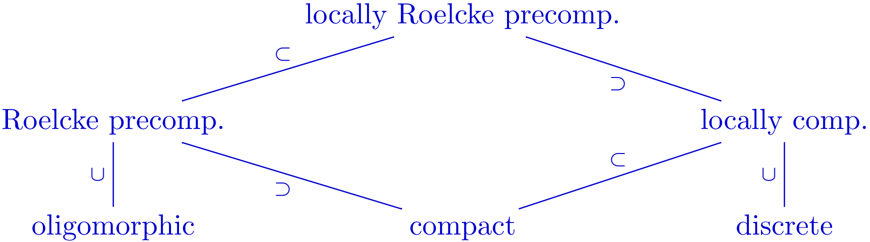
Definition

- G is Roelcke precompact (R.p.) if each open subgroup U has only finitely many double cosets.
- G is locally Roelcke precompact if G has a Roelcke precompact open subgroup.

Let T_∞ be the undirected tree with each vertex of infinite degree.

- $\mathrm{Aut}(T_\infty)$ is locally R.p. (Zielinski), and not locally compact.
- The stabiliser of a vertex is Roelcke precompact.

Some Borel classes $\mathcal{C}$, and inclusion relations



- Isomorphism relation on each class in the diagram is $\leq_B$ graph isomorphism (Kechris, N. and Tent, 2018).
- $\cong$ on the profinite groups is $\geq_B$ graph isomorphism (Kechris, N. and Tent, 2018).
- $\cong$ on the class of oligomorphic groups is $\leq_B$ a countable Borel equivalence relation (N., Schlicht and Tent, 2021).

Coarse group associated with a l.R.p. group

The following originiates in Kechris, N., Tent JSL 2018.

Given a locally R.p. G, let $\mathcal{M}(G)$ be its coarse group:

- The domain consists of (numbers encoding) the Roelcke precompact open cosets in G.
- Ternary relation "$AB \subseteq C$" on the domain.

R.p. open cosets approximate elements of G, so this ternary relation approximates the binary group operation.

Each R.p. open subgroup of G is a finite union of double cosets of a basic open subgroup. So $\exists$ only countably many such subgroups.

Using descriptive set theory, we can view the operator $\mathcal{M}$ as a Borel function from locally R.p. groups to structures with domain $\mathbb{N}$.

Define an operation $\mathcal{G}$ reversing $\mathcal{M}$: from coarse groups to locally R.p. groups

- Recall that $\mathcal{M}(G)$ is the coarse group of a locally R.p. G.
- Let **CG** be the closure under isomorphism of the range of $\mathcal{M}$, among the structures on $\mathbb{N}$ with a ternary relation.
- Write such a relation suggestively as "$AB \sqsubseteq C$".

Definition

Given a structure $M \in \mathbf{CG}$, let $\mathcal{G}(M)$ be the closed subgroup of $\mathrm{Sym}(\mathbb{N})$ consisting of the permutations p such that

$$AB \sqsubseteq C \iff p(A)B \sqsubseteq p(C) \text{ for each } A, B, C \in M.$$

Recall that we have defined maps $\mathbf{LRP} \underset{\mathcal{G}}{\overset{\mathcal{M}}{\rightleftarrows}} \mathbf{CG}$.

Note that $\mathcal{M}$ and $\mathcal{G}$ forward-preserve (topological) isomorphism.

Theorem (Borel duality for l.R.p. groups)

- $\mathbf{CG}$ is a Borel class. $\mathcal{M}$ and $\mathcal{G}$ are Borel maps.
- $\mathcal{M}$ and $\mathcal{G}$ are inverses up to isomorphism:
  For each $G \in \mathbf{LRP}$ and each $M \in \mathbf{CG}$,

$$\mathcal{G}(\mathcal{M}(G)) \cong_{top} G \text{ and } \mathcal{M}(\mathcal{G}(M)) \cong M.$$

As a consequence, for $G_0, G_1 \in \mathbf{LRP}$ and $M_0, M_1 \in \mathbf{CG}$, we have

$$G_0 \cong_{top} G_1 \iff \mathcal{M}(G_0) \cong \mathcal{M}(G_1)$$

$$M_0 \cong M_1 \iff \mathcal{G}(M_0) \cong_{top} \mathcal{G}(M_1)$$

The case of oligomorphic groups G

Note that G and hence each open subgroup is Roelcke precompact.

Theorem (NST, 21)

Among structures on $\mathbb{N}$ with a ternary relation symbol, let $\mathcal{D}$ be the closure under isomorphism of $\{\mathcal{M}(G) \colon G \text{ is oligomorphic}\}$.

(a) The class $\mathcal{D}$ is Borel.

(b) $\cong_{top}$ on the oligomorphic groups is Borel equivalent with the isomorphism relation on $\mathcal{D}$.

(a) is proved by a suitable axiomatisation of the class $\mathcal{D}$ using an infinitary language;
(b) is obtained via Borel duality for oligomorphic groups:

introduce a modification $\widehat{\mathcal{G}}$ of the "reverse" Borel operator $\mathcal{G}$ so that $\widehat{\mathcal{G}}(M)$ is oligomorphic for $M \in \mathcal{D}$.

A result of Hjorth and Kechris

The following will be used for showing that bi-interpretability on ω-categorical structures is $\leq_B E_\infty$ (a Borel equivalence relation with all classes countable):

Theorem (Hjorth and Kechris, APAL 1997, Th. 3.8)

- Let $\mathcal{D}$ be a Borel class of structures with domain $\mathbb{N}$.
- Suppose that $\cong_{\mathcal{D}}$ is potentially F_σ; that is, $\cong_{\mathcal{D}} \leq_B L$ for some F_σ equivalence relation L on some Polish space Y.

Then $\cong_{\mathcal{D}} \leq_B E_\infty$

In our setting $\mathcal{D}$ is the class of coarse groups above.
We will verify the hypothesis on $\mathcal{D}$ by showing that the relation of bi-interpretability among ω-categorical structures is F_σ.